# THE TOPOLOGICAL CENTERS OF MODULE ACTIONS

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Abstract. In this paper, we will study the topological centers of $n-th$ dual of Banach $A-module$ and we extend some propositions from Lau and $\ddot{U}lger$ into $n-th$ dual of Banach $A-modules$. Let $B$ be a Banach $A-bimodule$. By using some new conditions, we show that ${Z^\ell}_{A^{(n)}}(B^{(n)})=B^{(n)}$ and ${Z^\ell}_{B^{(n)}}(A^{(n)})=A^{(n)}$. We also have some conclusions in dual groups.

## 1. Preliminaries and Introduction

Throughout this paper, $A$ is a Banach algebra and $A^*$, $A^{**}$, respectively, are the first and second dual of $A$. Recall that a left approximate identity ($=LAI$) [resp. right approximate identity ($=RAI$)] in Banach algebra $A$ is a net $(e_\alpha)_{\alpha\in I}$ in $A$ such that $e_\alpha a\longrightarrow a$ [resp. $ae_\alpha\longrightarrow a$]. We say that a net $(e_\alpha)_{\alpha\in I}\subseteq A$ is a approximate identity ($=AI$) for $A$ if it is $LAI$ and $RAI$ for $A$. If $(e_\alpha)_{\alpha\in I}$ in $A$ is bounded and $AI$ for $A$, then we say that $(e_\alpha)_{\alpha\in I}$ is a bounded approximate identity ($=BAI$) for $A$. For $a\in A$ and $a'\in A^*$, we denote by $a'a$ and $aa'$ respectively, the functionals on $A^*$ defined by $<a'a,b>=<a',ab>=a'(ab)$ and $<aa',b>=<a',ba>=a'(ba)$ for all $b\in A$. The Banach algebra $A$ is embedded in its second dual via the identification $<a,a'>$ - $<a',a>$ for every $a\in A$ and $a'\in A^*$. We denote the set $\{a'a:\ a\in A\ and\ a'\in A^*\}$ and $\{aa':\ a\in A\ and\ a'\in A^*\}$ by $A^*A$ and $AA^*$, respectively, clearly these two sets are subsets of $A^*$.

Let $A$ have a $BAI$. If the equality $A^*A=A^*$, $(AA^*=A^*)$ holds, then we say that $A^*$ factors on the left (right). If both equalities $A^*A=AA^*=A^*$ hold, then we say that $A^*$ factors on both sides.

The extension of bilinear maps on normed space and the concept of regularity of bilinear maps were studied by [1, 2, 3, 6, 8, 14]. We start by recalling these definitions as follows.

Let $X,Y,Z$ be normed spaces and $m:X\times Y\rightarrow Z$ be a bounded bilinear mapping. Arens in [1] offers two natural extensions $m^{***}$ and $m^{t***t}$ of $m$ from $X^{**}\times Y^{**}$ into $Z^{**}$ as following

1. $m^*:Z^*\times X\rightarrow Y^*$, given by $<m^*(z',x),y>=<z',m(x,y)>$ where $x\in X$, $y\in Y$, $z'\in Z^*$,
2. $m^{**}:Y^{**}\times Z^*\rightarrow X^*$, given by $<m^{**}(y'',z'),x>=<y'',m^*(z',x)>$ where $x\in X$, $y''\in Y^{**}$, $z'\in Z^*$,
3. $m^{***}:X^{**}\times Y^{**}\rightarrow Z^{**}$, given by $<m^{***}(x'',y''),z'>$ $=<x'',m^{**}(y'',z')>$ where $x''\in X^{**}$, $y''\in Y^{**}$, $z'\in Z^*$.

The mapping $m^{***}$ is the unique extension of $m$ such that $x''\rightarrow m^{***}(x'',y'')$ from

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$X^{**}$ into $Z^{**}$ is $weak^* - weak^*$ continuous for every $y'' \in Y^{**}$, but the mapping $y'' \to m^{***}(x'', y'')$ is not in general $weak^* - weak^*$ continuous from $Y^{**}$ into $Z^{**}$ unless $x'' \in X$. Hence the first topological center of $m$ may be defined as following

$$Z_1(m) = \{x'' \in X^{**} : \ y'' \to m^{***}(x'', y'') \ is\ weak^* - weak^*\ continuous\}.$$

Let now $m^t : Y \times X \to Z$ be the transpose of $m$ defined by $m^t(y, x) = m(x, y)$ for every $x \in X$ and $y \in Y$. Then $m^t$ is a continuous bilinear map from $Y \times X$ to $Z$, and so it may be extended as above to $m^{t***} : Y^{**} \times X^{**} \to Z^{**}$. The mapping $m^{t***t} : X^{**} \times Y^{**} \to Z^{**}$ in general is not equal to $m^{***}$, see [1], if $m^{***} = m^{t***t}$, then $m$ is called Arens regular. The mapping $y'' \to m^{t***t}(x'', y'')$ is $weak^* - weak^*$ continuous for every $y'' \in Y^{**}$, but the mapping $x'' \to m^{t***t}(x'', y'')$ from $X^{**}$ into $Z^{**}$ is not in general $weak^* - weak^*$ continuous for every $y'' \in Y^{**}$. So we define the second topological center of $m$ as

$$Z_2(m) = \{y'' \in Y^{**} : \ x'' \to m^{t***t}(x'', y'') \ is\ weak^* - weak^*\ continuous\}.$$

It is clear that $m$ is Arens regular if and only if $Z_1(m) = X^{**}$ or $Z_2(m) = Y^{**}$. Arens regularity of $m$ is equivalent to the following

$$\lim_i \lim_j < z', m(x_i, y_j) >= \lim_j \lim_i < z', m(x_i, y_j) >,$$

whenever both limits exist for all bounded sequences $(x_i)_i \subseteq X$ , $(y_i)_i \subseteq Y$ and $z' \in Z^*$, see [18].
The mapping $m$ is left strongly Arens irregular if $Z_1(m) = X$ and $m$ is right strongly Arens irregular if $Z_2(m) = Y$.
Let now $B$ be a Banach $A - bimodule$, and let

$$\pi_\ell : \ A \times B \to B \ and\ \pi_r : \ B \times A \to B.$$

be the left and right module actions of $A$ on $B$, respectively. Then $B^{**}$ is a Banach $A^{**} - bimodule$ with module actions

$$\pi_\ell^{***} : \ A^{**} \times B^{**} \to B^{**} \ and\ \pi_r^{***} : \ B^{**} \times A^{**} \to B^{**}.$$

Similarly, $B^{**}$ is a Banach $A^{**} - bimodule$ with module actions

$$\pi_\ell^{t***t} : \ A^{**} \times B^{**} \to B^{**} \ and\ \pi_r^{t***t} : \ B^{**} \times A^{**} \to B^{**}.$$

We may therefore define the topological centers of the left and right module actions of $A$ on $B$ as follows:

$$\begin{aligned}
Z_{B^{**}}(A^{**}) = Z(\pi_\ell) = \{a'' \in A^{**} :\ & the\ map\ b'' \to \pi_\ell^{***}(a'', b'') \ : \ B^{**} \to B^{**} \\
& is\ weak^* - weak^*\ continuous\} \\
Z^t_{B^{**}}(A^{**}) = Z(\pi_r^t) = \{a'' \in A^{**} :\ & the\ map\ b'' \to \pi_r^{t***}(a'', b'') \ : \ B^{**} \to B^{**} \\
& is\ weak^* - weak^*\ continuous\} \\
Z_{A^{**}}(B^{**}) = Z(\pi_r) = \{b'' \in B^{**} :\ & the\ map\ a'' \to \pi_r^{***}(b'', a'') \ : \ A^{**} \to B^{**} \\
& is\ weak^* - weak^*\ continuous\} \\
Z^t_{A^{**}}(B^{**}) = Z(\pi_\ell^t) = \{b'' \in B^{**} :\ & the\ map\ a'' \to \pi_\ell^{t***}(b'', a'') \ : \ A^{**} \to B^{**} \\
& is\ weak^* - weak^*\ continuous\}
\end{aligned}$$

We note also that if $B$ is a left(resp. right) Banach $A-module$ and $\pi_\ell: A\times B\to B$ (resp. $\pi_r: B\times A\to B$) is left (resp. right) module action of $A$ on $B$, then $B^*$ is a right (resp. left) Banach $A-module$.
We write $ab=\pi_\ell(a,b)$, $ba=\pi_r(b,a)$, $\pi_\ell(a_1a_2,b)=\pi_\ell(a_1,a_2b)$,
$\pi_r(b,a_1a_2)=\pi_r(ba_1,a_2)$, $\pi_\ell^*(a_1b',a_2)=\pi_\ell^*(b',a_2a_1)$, $\pi_r^*(b'a,b)=\pi_r^*(b',ab)$,
for all $a_1,a_2,a\in A$, $b\in B$ and $b'\in B^*$ when there is no confusion.
Regarding $A$ as a Banach $A-bimodule$, the operation $\pi: A\times A\to A$ extends to $\pi^{***}$ and $\pi^{t***t}$ defined on $A^{**}\times A^{**}$. These extensions are known, respectively, as the first(left) and the second (right) Arens products, and with each of them, the second dual space $A^{**}$ becomes a Banach algebra. In this situation, we shall also simplify our notations. So the first (left) Arens product of $a'',b''\in A^{**}$ shall be simply indicated by $a''b''$ and defined by the three steps:

$$< a'a,b>=< a',ab>,$$
$$< a''a',a>=< a'',a'a>,$$
$$< a''b'',a'>=< a'',b''a'> .$$

for every $a,b\in A$ and $a'\in A^*$. Similarly, the second (right) Arens product of $a'',b''\in A^{**}$ shall be indicated by $a''ob''$ and defined by :

$$< aoa',b>=< a',ba>,$$
$$< a'oa'',a>=< a'',aoa'>,$$
$$< a''ob'',a'>=< b'',a'ob''> .$$

for all $a,b\in A$ and $a'\in A^*$.
The regularity of a normed algebra $A$ is defined to be the regularity of its algebra multiplication when considered as a bilinear mapping. Let $a''$ and $b''$ be elements of $A^{**}$, the second dual of $A$. By *Goldstine's* Theorem [4, P.424-425], there are nets $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in $A$ such that $a''=weak^*-\lim_\alpha a_\alpha$ and $b''=weak^*-\lim_\beta b_\beta$. So it is easy to see that for all $a'\in A^*$,

$$\lim_\alpha\lim_\beta < a',\pi(a_\alpha,b_\beta)>=< a''b'',a'>$$

and

$$\lim_\beta\lim_\alpha < a',\pi(a_\alpha,b_\beta)>=< a''ob'',a'>,$$

where $a''b''$ and $a''ob''$ are the first and second Arens products of $A^{**}$, respectively, see [14, 18].
We find the usual first and second topological center of $A^{**}$, which are

$$Z_{A^{**}}(A^{**})=Z(\pi)=\{a''\in A^{**}: b''\to a''b''\ is\ weak^*-weak^*\ continuous\},$$
$$Z^t_{A^{**}}(A^{**})=Z(\pi^t)=\{a''\in A^{**}: a''\to a''ob''\ is\ weak^*-weak^*\ continuous\}.$$

An element $e''$ of $A^{**}$ is said to be a mixed unit if $e''$ is a right unit for the first Arens multiplication and a left unit for the second Arens multiplication. That is, $e''$ is a mixed unit if and only if, for each $a''\in A^{**}$, $a''e''=e''oa''=a''$. By [4, p.146], an element $e''$ of $A^{**}$ is mixed unit if and only if it is a $weak^*$ cluster point of some BAI $(e_\alpha)_{\alpha\in I}$ in $A$.

A functional $a'$ in $A^*$ is said to be *wap* (weakly almost periodic) on $A$ if the mapping $a \rightarrow a'a$ from $A$ into $A^*$ is weakly compact. Pym in [18] showed that this definition to the equivalent following condition
For any two net $(a_\alpha)_\alpha$ and $(b_\beta)_\beta$ in $\{a \in A : \parallel a \parallel \leq 1\}$, we have

$$\lim_\alpha \lim_\beta < a', a_\alpha b_\beta >= \lim_\beta \lim_\alpha < a', a_\alpha b_\beta >,$$

whenever both iterated limits exist. The collection of all *wap* functionals on $A$ is denoted by $wap(A)$. Also we have $a' \in wap(A)$ if and only if $< a''b'', a' >=< a''ob'', a' >$ for every $a'', \ b'' \in A^{**}$.
This paper is organized as follows:
**a**) Let $B$ be a left Banach $A-bimodule$ and $\phi \in (A^{(n-r)}A^{(r)})^{(r)}$ where $0 \leq r \leq n$. Then $\phi \in {Z^\ell}_{B^{(n)}}((A^{(n-r)}A^{(r)})^{(r)})$ if and only if $b^{(n-1)}\phi \in B^{(n-1)}$ for all $b^{(n-1)} \in B^{(n-1)}$.
**b**) Let $B$ be a Banach $A-bimodule$. Then we have the following assertions.
  (1) $b^{(n)} \in {Z^\ell}_{A^{(n)}}(B^{(n)})$ if and only if $b^{(n-1)}b^{(n)} \in A^{(n-1)}$ for all $b^{(n-1)} \in B^{(n-1)}$.
  (2) If $b^{(n-1)}b^{(n)} \in A^{(n-1)}A^*$ for each $b^{(n-1)} \in B^{(n-1)}$, then we have $b^{(n)} \in {Z^\ell}_{A^{(n-1)}A^*}(B^{(n)})$.
  (3) If $\phi \in {Z^\ell}_{B^{(n)}}((A^{(n-r)}A^{(r)})^{(r)})$, then $a^{(n-1)}\phi \in {Z^\ell}_{B^{(n)}}(A^{(n)})$ for all $a^{(n-1)} \in A^{(n-1)}$.

**c**) Let $B$ be a Banach space such that $B^{(n)}$ is weakly compact. Then for Banach $A-bimodule$ $B$, we have the following assertions.
  (1) Suppose that $(e_\alpha^{(n)})_\alpha \subseteq A^{(n)}$ is a $BLAI$ for $B^{(n)}$ such that $e_\alpha^{(n)}B^{(n+2)} \subseteq B^{(n)}$ for every $\alpha$. Then $B$ is reflexive.
  (2) Suppose that $(e_\alpha^{(n)})_\alpha \subseteq A^{(n)}$ is a $BRAI$ for $B^{(n)}$ and $Z^\ell_{e^{(n+2)}}(B^{(n+2)}) = B^{(n+2)}$ where $e_\alpha^{(n)} \xrightarrow{w^*} e^{(n+2)}$ on $A^{(n)}$. If $B^{(n+2)}e_\alpha^{(n)} \subseteq B^{(n)}$ for every $\alpha$, then $Z^\ell_{A^{(n+2)}}(B^{(n+2)}) = B^{(n+2)}$.

**d**) Assume that $B$ is a Banach $A-bimodule$. Then we have the following assertions.
  (1) $B^{(n+1)}A^{(n)} \subseteq wap_\ell(B^{(n)})$ if and only if $A^{(n)}A^{(n+2)} \subseteq Z^\ell_{B^{(n+2)}}(A^{(n+2)})$.
  (2) If $A^{(n)}A^{(n+2)} \subseteq A^{(n)}{Z^\ell}_{B^{(n+2)}}(A^{(n+2)})$, then

$$A^{(n)}A^{(n+2)} \subseteq Z^\ell_{B^{(n+2)}}(A^{(n+2)}).$$

**e**) Let $B$ be a left Banach $A-bimodule$ and $n \geq 0$ be a even. Suppose that $b_0^{(n+1)} \in B^{(n+1)}$. Then $b_0^{(n+1)} \in wap_\ell(B^{(n)})$ if and only if the mapping $T : b^{(n+2)} \rightarrow b^{(n+2)}b_0^{(n+1)}$ form $B^{(n+2)}$ into $A^{(n+1)}$ is $weak^*-to-weak$ continuous.
**f**) Let $B$ be a left Banach $A-bimodule$. Then for $n \geq 2$, we have the following assertions.
  (1) If $A^{(n)} = a_0^{(n-2)}A^{(n)}$ [resp. $A^{(n)} = A^{(n)}a_0^{(n-2)}$] for some $a_0^{(n-2)} \in A^{(n-2)}$ and $a_0^{(n-2)}$ has $Rw^*w-$ property [resp. $Lw^*w-$ property] with respect to $B^{(n)}$, then $Z_{B^{(n)}}(A^{(n)}) = A^{(n)}$.

  (2) If $B^{(n)} = a_0^{(n-2)}B^{(n)}$ [resp. $B^{(n)} = B^{(n)}a_0^{(n-2)}$] for some $a_0^{(n-2)} \in A^{(n-2)}$ and $a_0^{(n-2)}$ has $Rw^*w-$ property [resp. $Lw^*w-$ property] with respect to $B^{(n)}$, then $Z_{A^{(n)}}(B^{(n)}) = B^{(n)}$.

## 2. Topological centers of module actions

Suppose that $A$ is a Banach algebra and $B$ is a Banach $A-bimodule$. According to [5, pp.27 and 28], $B^{**}$ is a Banach $A^{**}-bimodule$, where $A^{**}$ is equipped with the first Arens product. So we recalled the topological centers of module actions as in the following.

$$Z^{\ell}_{A^{**}}(B^{**})=\{b''\in B^{**}:\ the\ map\ a''\to b''a''\ :\ A^{**}\to B^{**}$$
$$is\ weak^*-weak^*\ continuous\}$$
$$Z^{\ell}_{B^{**}}(A^{**})=\{a''\in A^{**}:\ the\ map\ b''\to a''b''\ :\ B^{**}\to B^{**}$$
$$is\ weak^*-weak^*\ continuous\}.$$

Let $A^{(n)}$ and $B^{(n)}$ be $n-th\ dual$ of $B$ and $A$, respectively. By [24], $B^{(n)}$ is a Banach $A^{(n)}-bimodule$ and we define $B^{(n)}B^{(n-1)}$ as a subspace of $A^{(n)}$, that is, for all $b^{(n)}\in B^{(n)}$ and $b^{(n-1)}\in B^{(n-1)}$, we define

$$<b^{(n)}b^{(n-1)},a^{(n-1)}>=<b^{(n)},b^{(n-1)}a^{(n-1)}>;$$

and if $n=0$, we take $A^{(0)}=A$ and $B^{(0)}=B$.

**Theorem 2-1.** Let $B$ be a left Banach $A-bimodule$ and $\phi\in(A^{(n-r)}A^{(r)})^{(r)}$ where $0\leq r\leq n$. Then $\phi\in {Z^{\ell}}_{B^{(n)}}((A^{(n-r)}A^{(r)})^{(r)})$ if and only if $b^{(n-1)}\phi\in B^{(n-1)}$ for all $b^{(n-1)}\in B^{(n-1)}$.

*Proof.* Let $\phi\in {Z^{\ell}}_{B^{(n)}}((A^{(n-r)}A^{(r)})^{(r)})$. Suppose that $(b_\alpha^{(n)})_\alpha\subseteq B^{(n)}$ such that $b_\alpha^{(n)}\overset{w^*}{\to}b^{(n)}$ on $B^{(n)}$. Then, for every $b^{(n-1)}\in B^{(n-1)}$, we have

$$<b^{(n-1)}\phi,b_\alpha^{(n)}>=<b^{(n-1)},\phi b_\alpha^{(n)}>=<\phi b_\alpha^{(n)},b^{(n-1)}>\to<\phi b^{(n)},b^{(n-1)}>$$
$$=<b^{(n-1)}\phi,b^{(n)}>.$$

It follows that $b^{(n-1)}\phi\in(B^{(n+1)},weak^*)^*=B^{(n-1)}$.

Conversely, let $b^{(n-1)}\phi\in B^{(n-1)}$ for every $b^{(n-1)}\in B^{(n-1)}$ and suppose that $(b_\alpha^{(n)})_\alpha\subseteq B^{(n)}$ such that $b_\alpha^{(n)}\overset{w^*}{\to}b^{(n)}$ on $B^{(n)}$. Then

$$<\phi b_\alpha^{(n)},b^{(n-1)}>=<\phi,b_\alpha^{(n)}b^{(n-1)}>=<b_\alpha^{(n)}b^{(n-1)},\phi>=<b_\alpha^{(n)},b^{(n-1)}\phi>$$
$$\to<b^{(n)},b^{(n-1)}\phi>=<\phi b^{(n)},b^{(n-1)}>.$$

It follows that $<\phi b_\alpha^{(n)}\overset{w^*}{\to}\phi b^{(n)}$, and so $\phi\in {Z^{\ell}}_{B^{(n)}}((A^{(n-r)}A^{(r)})^{(r)})$. □

Let $B$ be a left Banach $A-bimodule$ and $\phi\in(A^{(n-r)}A^{(r)})^{(n-r)}$ where $0\leq r\leq n$. Then $\phi\in {Z^{\ell}}_{B^{(n)}}((A^{(n-r)}A^{(r)})^{(n-r)})$ if and only if $b^{(n-1)}\phi\in B^{(n-1)}$ for every $b^{(n-1)}\in B^{(n-1)}$. The proof of the proceeding assertion is the similar to Theorem 2-1, and if we take $B=A$, $n=1$ and $r=0$, we obtain Lemma 3.1 (b) from [14].

**Theorem 2-2.** Let $B$ be a Banach $A-bimodule$. Then we have the following assertions.

(1) $b^{(n)}\in {Z^{\ell}}_{A^{(n)}}(B^{(n)})$ if and only if $b^{(n-1)}b^{(n)}\in A^{(n-1)}$ for all $b^{(n-1)}\in B^{(n-1)}$.

(2) If $b^{(n-1)}b^{(n)} \in A^{(n-1)}A^*$ for each $b^{(n-1)} \in B^{(n-1)}$, then we have $b^{(n)} \in Z^{\ell}{}_{A^{(n-1)}A^*}(B^{(n)})$.

(3) If $\phi \in Z^{\ell}{}_{B^{(n)}}((A^{(n-r)}A^{(r)})^{(r)})$, then $a^{(n-1)}\phi \in Z^{\ell}{}_{B^{(n)}}(A^{(n)})$ for all $a^{(n-1)} \in A^{(n-1)}$.

*Proof.* (1) Let $b^{(n)} \in Z^{\ell}{}_{A^{(n)}}(B^{(n)})$. We show that $b^{(n-1)}b^{(n)} \in A^{(n-1)}$ where $b^{(n-1)} \in B^{(n-1)}$. Suppose that $(a_\alpha^{(n)})_\alpha \subseteq A^{(n)}$ and $a_\alpha^{(n)} \xrightarrow{w^*} a^{(n)}$ on $A^{(n)}$. Then we have

$$< b^{(n-1)}b^{(n)}, a_\alpha^{(n)} >=< b^{(n-1)}, b^{(n)}a_\alpha^{(n)} >=< b^{(n)}a_\alpha^{(n)}, b^{(n-1)} >$$

$$\rightarrow< b^{(n)}a^{(n)}, b^{(n-1)} >=< b^{(n-1)}b^{(n)}, a^{(n)} > .$$

Consequently $b^{(n-1)}b^{(n)} \in (A^{(n+1)}, weak^*)^* = A^{(n-1)}$. It follows that $b^{(n-1)}b^{(n)} \in A^{(n-1)}$.

Conversely, let $b^{(n-1)}b^{(n)} \in A^{(n-1)}$ for each $b^{(n-1)} \in B^{(n-1)}$. Suppose that $(a_\alpha^{(n)})_\alpha \subseteq A^{(n)}$ and $a_\alpha^{(n)} \xrightarrow{w^*} a^{(n)}$ on $A^{(n)}$. Then we have

$$< b^{(n)}a_\alpha^{(n)}, b^{(n-1)} >=< b^{(n)}, a_\alpha^{(n)}b^{(n-1)} >=< a_\alpha^{(n)}b^{(n-1)}, b^{(n)} >$$

$$=< a_\alpha^{(n)}, b^{(n-1)}b^{(n)} >\rightarrow< a^{(n)}, b^{(n-1)}b^{(n)} >=< b^{(n)}a^{(n)}, b^{(n-1)} > .$$

It follows that $b^{(n)}a_\alpha^{(n)} \xrightarrow{w^*} b^{(n)}a^{(n)}$, and so $b^{(n)} \in Z^{\ell}{}_{A^{(n)}}(B^{(n)})$.

(2) Proof is similar to (1).

(3) Let $\phi \in Z^{\ell}{}_{B^{(n)}}((A^{(n-r)}A^{(r)})^{(r)})$ and $a^{(n-1)} \in A^{(n-1)}$. Assume that $(b_\alpha^{(n)})_\alpha \subseteq B^{(n)}$ and $b_\alpha^{(n)} \xrightarrow{w^*} b^{(n)}$ on $B^{(n)}$. Then for all $b^{(n-1)} \in B^{(n-1)}$, we have

$$< (a^{(n-1)}\phi)b_\alpha^{(n)}, b^{(n-1)} >=< \phi b_\alpha^{(n)}, b^{(n-1)}a^{(n-1)} >\rightarrow< \phi b^{(n)}, b^{(n-1)}a^{(n-1)} >$$

$$=< (a^{(n-1)}\phi)b^{(n)}, b^{(n-1)} > .$$

It follows that $(a^{(n-1)}\phi)b_\alpha^{(n)} \xrightarrow{w^*} (a^{(n-1)}\phi)b^{(n)}$, and so $a^{(n-1)}\phi \in Z^{\ell}{}_{B^{(n)}}(A^{(n)})$. $\square$

In the proceeding theorem, part (1), if we take $B = A$ and $n = 2$, we conclude Lemma 3.1 (a) from [14]. We change part (3) of Theorem 2-2 with the following statement, that is, if $\phi \in Z^{\ell}{}_{B^{(n)}}((A^{(n-r)}A^{(r)})^{(n-r)})$, then $a^{(n-1)}\phi \in Z^{\ell}{}_{B^{(n)}}(A^{(n)})$ for all $a^{(n-1)} \in A^{(n-1)}$. Proof of this claim is similar to proceeding theorem and if we take $B = A$, $n = 1$ and $r = 0$, we obtain Lemma 3.1 (c) from [14].

**Definition 2-3.** Let $B$ be a Banach $A-bimodule$ and suppose that $a'' \in A^{**}$. We say that $a'' \rightarrow b''a''$ is $weak^* - to - weak^*$ point continuous, if for every net $(a''_\alpha)_\alpha \subseteq A^{**}$ such that $a''_\alpha \xrightarrow{w^*} a''$, it follows that $a''_\alpha b'' \xrightarrow{w^*} a''b''$.

Suppose that $B$ is a Banach $A - bimodule$. Assume that $a'' \in A^{**}$. Then we define the locally topological center of $a''$ on $B^{**}$ as follows

$$Z^{\ell}_{a''}(B^{**}) = \{b'' \in B^{**} : \ a'' \rightarrow b''a'' \ is\ weak^* - to - weak^* \ point\ continuous\},$$

The definition of $Z^{\ell}_{b''}(A^{**})$ where $b'' \in B^{**}$ are similar.

It is clear that

$$\bigcap_{a'' \in A^{**}} Z^{\ell}_{a''}(B^{**}) = Z^{\ell}_{A^{**}}(B^{**}),$$

$$\bigcap_{b''\in B^{**}} Z^{\ell}_{b''}(A^{**}) = Z^{\ell}_{B^{**}}(A^{**}).$$

**Theorem 2-4.** Let $B$ be a Banach space such that $B^{(n)}$ is weakly compact. Then for Banach $A-bimodule$ $B$, we have the following assertions.

(1) Suppose that $(e_\alpha^{(n)})_\alpha \subseteq A^{(n)}$ is a $BLAI$ for $B^{(n)}$ such that $e_\alpha^{(n)}B^{(n+2)} \subseteq B^{(n)}$ for every $\alpha$. Then $B$ is reflexive.

(2) Suppose that $(e_\alpha^{(n)})_\alpha \subseteq A^{(n)}$ is a $BRAI$ for $B^{(n)}$ and $Z^{\ell}_{e^{(n+2)}}(B^{(n+2)}) = B^{(n+2)}$ where $e_\alpha^{(n)} \overset{w^*}{\to} e^{(n+2)}$ on $A^{(n)}$. If $B^{(n+2)}e_\alpha^{(n)} \subseteq B^{(n)}$ for every $\alpha$, then $Z^{\ell}_{A^{(n+2)}}(B^{(n+2)}) = B^{(n+2)}$.

*Proof.* (1) Let $b^{n+2} \in B^{n+2}$. Since $(e_\alpha^{(n)})_\alpha$ is a $BLAI$ for $B^{(n)}$, there is left unit as $e^{(n+2)} \in A^{n+2}$ for $B^{n+2}$, see [10]. Then we have $e_\alpha^{(n)}b^{(n+2)} \overset{w^*}{\to} b^{(n+2)}$ on $B^{(n+2)}$. Since $e_\alpha^{(n)}b^{(n+2)} \in B^{(n)}$, we have $e_\alpha^{(n)}b^{(n+2)} \overset{w}{\to} b^{(n+2)}$ on $B^{(n)}$. We conclude that $b^{n+2} \in B^n$ of course $B^n$ is weakly compact.

(2) Suppose that $b^{(n+2)} \in Z^{\ell}_{A^{(n+2)}}(B^{(n+2)})$ and $e_\alpha^{(n)} \overset{w^*}{\to} e^{(n+2)}$ on $A^{(n)}$ such that $e^{(n+2)}$ is right unit for $B^{(n+2)}$, see [10]. Then we have $b^{(n+2)}e_\alpha^{(n)} \overset{w^*}{\to} b^{(n+2)}$ on $B^{(n+2)}$. Since $B^{(n+2)}e_\alpha^{(n)} \subseteq B^{(n)}$ for every $\alpha$, $b^{(n+2)}e_\alpha^{(n)} \overset{w}{\to} b^{(n+2)}$ on $B^{(n)}$ and since $B^{(n)}$ is weakly compact, $b^{(n+2)} \in B^{(n)}$. It follows that $Z^{\ell}_{A^{(n+2)}}(B^{(n+2)}) = B^{(n+2)}$.

□

**Definition 2-5.** Let $B$ be a Banach $A-bimodule$. Then $b^{n+1} \in B^{n+1}$ is said to be weakly left almost periodic functional if the set

$$\{b^{(n+1)}a^{(n)} :\ a^{(n)} \in A^{(n)},\ \| a^{(n)} \|\leq 1\},$$

is relatively weakly compact, and $b^{n+1} \in B^{n+1}$ is said to be weakly right almost periodic functional if the set

$$\{a^{(n)}b^{(n+1)} :\ a^{(n)} \in A^{(n)},\ \| a^{(n)} \|\leq 1\},$$

is relatively weakly compact. We denote by $wap_\ell(B^{(n)})$ [resp. $wap_\ell(B^{(n)})$] the closed subspace of $B^{(n+1)}$ consisting of all the weakly left [resp. right] almost periodic functionals in $B^{(n+1)}$. By [6, 14, 18], the definition of $wap_\ell(B^{(n)})$ and $wap_r(B^{(n)})$, respectively, are equivalent to the following

$$wap_\ell(B^{(n)}) = \{b^{(n+1)} \in B^{(n+1)} : < b^{(n+2)}a_\alpha^{(n+2)}, b^{(n+1)} >\to< b^{(n+2)}a^{(n+2)}, b^{(n+1)} >$$

$$where\ a_\alpha^{(n+2)} \overset{w^*}{\to} a^{(n+2)}\}.$$

and

$$wap_r(B^{(n)}) = \{b^{(n+1)} \in B^{(n+1)} : < a^{(n+2)}b_\alpha^{(n+2)}, b^{(n+1)} >\to< a^{(n+2)}b^{(n+2)}, b^{(n+1)} >$$

$$where\ b_\alpha^{(n+2)} \overset{w^*}{\to} b^{(n+2)}\}.$$

If we take $A = B$ and $n = 0$, then $wap_\ell(A) = wap_r(A) = wap(A)$.

**Theorem 2-6.** Assume that $B$ is a Banach $A-bimodule$. Then we have the following assertions.

(1) $B^{(n+1)}A^{(n)} \subseteq wap_\ell(B^{(n)})$ if and only if $A^{(n)}A^{(n+2)} \subseteq Z^\ell_{B^{(n+2)}}(A^{(n+2)})$.

(2) If $A^{(n)}A^{(n+2)} \subseteq A^{(n)}Z^\ell{}_{B^{(n+2)}}(A^{(n+2)})$, then
$$A^{(n)}A^{(n+2)} \subseteq Z^\ell_{B^{(n+2)}}(A^{(n+2)}).$$

*Proof.* (1) Suppose that $B^{(n+1)}A^{(n)} \subseteq wap_\ell(B^{(n)})$. Let $a^{(n)} \in A^{(n)}$, $a^{(n+2)} \in A^{(n+2)}$ and let $(b_\alpha^{(n+2)})_\alpha \subseteq B^{(n+2)}$ such that $b_\alpha^{(n+2)} \stackrel{w^*}{\to} b^{(n+2)}$. Then for every $b^{(n+1)} \in B^{(n+1)}$, we have
$$< (a^{(n)}a^{(n+2)})b_\alpha^{(n+2)}, b^{(n+1)} >=< a^{(n+2)}b_\alpha^{(n+2)}, b^{(n+1)}a^{(n)} >$$
$$\to< a^{(n+2)}b^{(n+2)}, b^{(n+1)}a^{(n)} >=< (a^{(n)}a^{(n+2)})b^{(n+2)}, b^{(n+1)} > .$$
It follows that $a^{(n)}a^{(n+2)} \in Z^\ell_{B^{(n+2)}}(A^{(n+2)})$.
Conversely, let $a^{(n)}a^{(n+2)} \in Z^\ell_{B^{(n+2)}}(A^{(n+2)})$ for every $a^{(n)} \in A^{(n)}$, $a^{(n+2)} \in A^{(n+2)}$ and suppose that $(b_\alpha^{(n+2)})_\alpha \subseteq B^{(n+2)}$ such that $b_\alpha^{(n+2)} \stackrel{w^*}{\to} b^{(n+2)}$. Then for every $b^{(n+1)} \in B^{(n+1)}$, we have
$$< a^{(n+2)}b_\alpha^{(n+2)}, b^{(n+1)}a^{(n)} >=< (a^{(n)}a^{(n+2)})b_\alpha^{(n+2)}, b^{(n+1)} >$$
$$\to< (a^{(n)}a^{(n+2)})b^{(n+2)}, b^{(n+1)} >=< a^{(n+2)}b_\alpha^{(n+2)}, b^{(n+1)}a^{(n)} > .$$
It follows that $B^{(n+1)}A^{(n)} \subseteq wap_\ell(B^{(n)})$.

(2) Since $A^{(n)}A^{(n+2)} \subseteq A^{(n)}Z^\ell{}_{B^{(n)}}((A^{(n+2)})$, for every $a^{(n)} \in A^{(n)}$ and $a^{(n+2)} \in A^{(n+2)}$, we have $a^{(n)}a^{(n+2)} \in A^{(n)}Z^\ell{}_{B^{(n+2)}}(A^{(n+2)})$.
Then there are $x^{(n)} \in A^{(n)}$ and $\phi \in Z^\ell_{B^{(n+2)}}(A^{(n+2)})$ such that $a^{(n)}a^{(n+2)} = x^{(n)}\phi$. Suppose that $(b_\alpha^{(n+2)})_\alpha \subseteq B^{(n+2)}$ such that $b_\alpha^{(n+2)} \stackrel{w^*}{\to} b^{(n+2)}$. Then for every $b^{(n+1)} \in B^{(n+1)}$, we have
$$< (a^{(n)}a^{(n+2)})b_\alpha^{(n+2)}, b^{(n+1)} >=< (x^{(n)}\phi)b_\alpha^{(n+2)}, b^{(n+1)} >$$
$$=< \phi b_\alpha^{(n+2)}, b^{(n+1)}x^{(n)} >\to< \phi b^{(n+2)}, b^{(n+1)}x^{(n)} >$$
$$=< (a^{(n)}a^{(n+2)})b^{(n+2)}, b^{(n+1)} > .$$
□

If we take $B = A$ and $n = 0$, we conclude Theorem 3.6 (a) from [14].

**Theorem 2-7.** Assume that $B$ is a Banach $A - bimodule$. If $A^{(n)}$ is a left ideal in $A^{(n+2)}$, then $B^{(n+1)}A^{(n)} \subseteq wap_\ell(B^{(n)})$.

**Theorem 2-8.** Let $B$ be a left Banach $A - bimodule$ and $n \geq 0$ be a even. Suppose that $b_0^{(n+1)} \in B^{(n+1)}$. Then $b_0^{(n+1)} \in wap_\ell(B^{(n)})$ if and only if the mapping $T : b^{(n+2)} \to b^{(n+2)}b_0^{(n+1)}$ form $B^{(n+2)}$ into $A^{(n+1)}$ is $weak^* - to - weak$ continuous.

*Proof.* Let $b_0^{(n+1)} \in B^{(n+1)}$ and suppose that $b_\alpha^{(n+2)} \stackrel{w^*}{\to} b^{(n+2)}$ on $B^{(n+2)}$. Then for every $a^{(n+2)} \in A^{(n+2)}$, we have
$$< a^{(n+2)}, b_\alpha^{(n+2)}b_0^{(n+1)} >=< a^{(n+2)}b_\alpha^{(n+2)}, b_0^{(n+1)} >\to< a^{(n+2)}b^{(n+2)}, b_0^{(n+1)} >$$

$$=< a^{(n+2)}, b^{(n+2)}b_0^{(n+1)} > .$$

It follows that $b_\alpha^{(n+2)}b_0^{(n+1)} \overset{w}{\to} b^{(n+2)}b_0^{(n+1)}$ on $A^{(n+1)}$.
The proof of the converse is similar of proceeding proof. $\square$

**Corollary 2-9.** Assume that $B$ is a Banach $A-bimodule$. Then $Z^{\ell}_{A^{(n+2)}}(B^{(n+2)}) = B^{(n+2)}$ if and only if the mapping $T : b^{(n+2)} \to b^{(n+2)}b_0^{(n+1)}$ form $B^{(n+2)}$ into $A^{(n+1)}$ is $weak^* - to - weak$ continuous for every $b_0^{(n+1)} \in B^{(n+1)}$.

**Corollary 2-10.** Let $A$ be a Banach algebra. Assume that $a' \in A^*$ and $T_{a'}$ is the linear operator from $A$ into $A^*$ defined by $T_{a'}a = a'a$. Then, $a' \in wap(A)$ if and only if the adjoint of $T_{a'}$ is $weak^* - to - weak$ continuous. So $A$ is Arens regular if and only if the adjoint of the mapping $T_{a'}a = a'a$ is $weak^* - to - weak$ continuous for every $a' \in A^*$.

**Definition 2-11.** Let $B$ be a left Banach $A-bimodule$. We say that $a^{(n)} \in A^{(n)}$ has $Left-weak^*-weak$ property ($= Lw^*w-$ property) with respect to $B^{(n)}$, if for every $(b_\alpha^{(n+1)})_\alpha \subseteq B^{(n+1)}$, $a^{(n)}b_\alpha^{(n+1)} \overset{w^*}{\to} 0$ implies $a^{(n)}b_\alpha^{(n+1)} \overset{w}{\to} 0$. If every $a^{(n)} \in A$ has $Lw^*w-$ property with respect to $B^{(n)}$, then we say that $A^{(n)}$ has $Lw^*w-$ property with respect to $B^{(n)}$. The definition of the $Right-weak^*-weak$ property ($= Rw^*w-$ property) is the same.
We say that $a^{(n)} \in A^{(n)}$ has $weak^*-weak$ property ($= w^*w-$ property) with respect to $B^{(n)}$ if it has $Lw^*w-$ property and $Rw^*w-$ property with respect to $B^{(n)}$.
If $a^{(n)} \in A^{(n)}$ has $Lw^*w-$ property with respect to itself, then we say that $a^{(n)} \in A^{(n)}$ has $Lw^*w-$ property.

**Example 2-12.**

(1) If $B$ is Banach $A$-bimodule and reflexive, then $A$ has $w^*w-$property with respect to $B$.
(2) $L^1(G)$, $M(G)$ and $A(G)$ have $w^*w-$property when $G$ is finite.
(3) Let $G$ be locally compact group. $L^1(G)$ [resp. $M(G)$] has $w^*w-$property [resp. $Lw^*w-$ property ]with respect to $L^p(G)$ whenever $p > 1$.
(4) Suppose that $B$ is a left Banach $A - module$ and $e$ is left unit element of $A$ such that $eb = b$ for all $b \in B$. If $e$ has $Lw^*w-$ property, then $B$ is reflexive.
(5) If $S$ is a compact semigroup, then $C^+(S) = \{f \in C(S) : \ f > 0\}$ has $w^*w-$property.

**Theorem 2-13.** Let $B$ be a left Banach $A - bimodule$. Then for $n \geq 2$, we have the following assertions.

(1) If $A^{(n)} = a_0^{(n-2)}A^{(n)}$ [resp. $A^{(n)} = A^{(n)}a_0^{(n-2)}$] for some $a_0^{(n-2)} \in A^{(n-2)}$ and $a_0^{(n-2)}$ has $Rw^*w-$ property [resp. $Lw^*w-$ property] with respect to $B^{(n)}$, then $Z_{B^{(n)}}(A^{(n)}) = A^{(n)}$.

(2) If $B^{(n)}=a_0^{(n-2)}B^{(n)}$ [resp. $B^{(n)}=B^{(n)}a_0^{(n-2)}$] for some $a_0^{(n-2)}\in A^{(n-2)}$ and $a_0^{(n-2)}$ has $Rw^*w-$ property [resp. $Lw^*w-$ property] with respect to $B^{(n)}$, then $Z_{A^{(n)}}(B^{(n)})=B^{(n)}$.

*Proof.* (1) Suppose that $A^{(n)}=a_0^{(n-2)}A^{(n)}$ for some $a_0^{(n-2)}\in A$ and $a_0^{(n-2)}$ has $Rw^*w-$ property. Let $(b_\alpha^{(n)})_\alpha\subseteq B^{(n)}$ such that $b_\alpha^{(n)}\stackrel{w^*}{\rightarrow}b^{(n)}$. Then for every $a^{(n-2)}\in A^{(n-2)}$ and $b^{(n-1)}\in B^{(n-1)}$, we have

$$< b_\alpha^{(n)}b^{(n-1)},a^{(n-2)}>=< b_\alpha^{(n)},b^{(n-1)}a^{(n-2)}>\rightarrow< b^{(n)},b^{(n-1)}a^{(n-2)}>$$

$$=< b^{(n)}b^{(n-1)},a^{(n-2)}>.$$

It follows that $b_\alpha^{(n)}b^{(n-1)}\stackrel{w^*}{\rightarrow}b^{(n)}b^{(n-1)}$. Also it is clear that $(b_\alpha^{(n)}b^{(n-1)})a_0^{(n-2)}\stackrel{w^*}{\rightarrow}(b^{(n)}b^{(n-1)})a_0^{(n-2)}$. Since $a_0^{(n-2)}$ has $Rw^*w-$ property, $(b_\alpha^{(n)}b^{(n-1)})a_0^{(n-2)}\stackrel{w}{\rightarrow}(b^{(n)}b^{(n-1)})a_0^{(n-2)}$. Now, let $a^{(n)}\in A^{(n)}$. Since $A^{(n)}=a_0^{(n-2)}A^{(n)}$, there is $x^{(n)}\in A^{(n)}$ such that $a^{(n)}=a_0^{(n-2)}x^{(n)}$. Thus we have

$$< a^{(n)}b_\alpha^{(n)},b^{(n-1)}>=< a^{(n)},b_\alpha^{(n)}b^{(n-1)}>=< a_0^{(n-2)}x^{(n)},b_\alpha^{(n)}b^{(n-1)}>$$

$$=< x^{(n)},(b_\alpha^{(n)}b^{(n-1)})a_0^{(n-2)}>\rightarrow< x^{(n)},(b^{(n)}b^{(n-1)})a_0^{(n-2)}>$$

$$=< a^{(n)}b,b^{(n-1)}>.$$

It follows that $a^{(n)}\in Z_{A^{(n)}}(B^{(n)})$.
Proof of the next part is similar to proceeding proof.

(2) Let $B^{(n)}=a_0^{(n-2)}B^{(n)}$ for some $a_0^{(n-2)}\in A$ and $a_0^{(n-2)}$ has $Rw^*w-$ property with respect to $B^{(n)}$. Assume that $(a_\alpha^{(n)})_\alpha\subseteq A^{(n)}$ such that $a_\alpha^{(n)}\stackrel{w^*}{\rightarrow}a^{(n)}$. Then for every $b^{(n-1)}\in B^{(n-1)}$, we have

$$< a_\alpha^{(n)}b^{(n-1)},b^{(n-2)}>=< a_\alpha^{(n)},b^{(n-1)}b^{(n-2)}>\rightarrow< a^{(n)},b^{(n-1)}b^{(n-2)}>$$

$$=< a^{(n)}b^{(n-1)},b^{(n-2)}>$$

We conclude that $a_\alpha^{(n)}b^{(n-1)}\stackrel{w^*}{\rightarrow}a^{(n)}b^{(n-1)}$. It is clear that $(a_\alpha^{(n)}b^{(n-1)})a_0^{(n-2)}\stackrel{w^*}{\rightarrow}(a^{(n)}b^{(n-1)})a_0^{(n-2)}$. Since $a_0^{(n-2)}$ has $Rw^*w-$ property, $(a_\alpha^{(n)}b^{(n-1)})a_0^{(n-2)}\stackrel{w}{\rightarrow}(a^{(n)}b^{(n-1)})a_0^{(n-2)}$.
Suppose that $b^{(n)}\in B^{(n)}$. Since $B^{(n)}=a_0^{(n-2)}B^{(n)}$, there is $y^{(n)}\in B^{(n)}$ such that $b^{(n)}=a_0^{(n-2)}y^{(n)}$. Consequently, we have

$$< b^{(n)}a_\alpha^{(n)},b^{(n-1)}>=< b^{(n)},a_\alpha^{(n)}b^{(n-1)}>=< a_0^{(n-2)}y^{(n)},a_\alpha^{(n)}b^{(n-1)}>$$

$$=< y^{(n)},(a_\alpha^{(n)}b^{(n-1)})a_0^{(n-2)}>\rightarrow< y^{(n)},(a^{(n)}b^{(n-1)})a_0^{(n-2)}>$$

$$=< a_0^{(n-2)}y^{(n)},(a^{(n)}b^{(n-1)})>=< b^{(n)}a^{(n)},b^{(n-1)}>.$$

Thus $b^{(n)}a_\alpha^{(n)}\stackrel{w}{\rightarrow}b^{(n)}a^{(n)}$. It follows that $b^{(n)}\in Z_{A^{(n)}}(B^{(n)})$.
The next part similar to the proceeding proof.

□

**Example 2-14.** Let $G$ be a locally compact group. Since $M(G)$ is a Banach $L^1(G)$-bimodule and the unit element of $M(G)^{(n)}$ has not $Lw^*w-$ property or $Rw^*w-$ property, by Theorem 2-13, $Z_{L^1(G)^{(n)}}(M(G)^{(n)}) \neq M(G)^{(n)}$.
ii) If $G$ is finite, then by Theorem 2-13, we have $Z_{M(G)^{(n)}}(L^1(G)^{(n)}) = L^1(G)^{(n)}$ and $Z_{L^1(G)^{(n)}}(M(G)^{(n)}) = M(G)^{(n)}$.

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